湖南大學
HUNAN UNIVERSITY

ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ
UNIVERSITY OF CRETE

Spectrality of a measure consisting of two line segments

Sha Wu

Join work with Mihalis Kolountzakis

Visiting University of Crete

2025.09

Xidian University, Xi'an, Shaanxi

Outline

- **Background**
- **Spectral problem**
    - Symmetric additive measures
    - Main results

Background

Classical example: $L^2([0,1])$ has an orthogonal basis $\left\{e^{2\pi i\lambda\cdot x}\right\}_{\lambda\in\mathbb{Z}}$,

i.e., for any $f\in L^2([0,1])$

$$f(x)=\sum_{\lambda\in\mathbb{Z}}c_\lambda(f)e_\lambda(x),\ \text{with } e_\lambda(x)=e^{2\pi i\lambda\cdot x},$$

where $c_\lambda(f)=\int_0^1 f(x)e^{-2\pi i\lambda\cdot x}\,dx$.

Background

Classical example: $L^2([0,1])$ has an orthogonal basis $\left\{e^{2\pi i\lambda\cdot x}\right\}_{\lambda\in\mathbb{Z}}$,

i.e., for any $f\in L^2([0,1])$

$$f(x)=\sum_{\lambda\in\mathbb{Z}}c_\lambda(f)e_\lambda(x),\ \text{with } e_\lambda(x)=e^{2\pi i\lambda\cdot x},$$

where $c_\lambda(f)=\int_0^1 f(x)e^{-2\pi i\lambda\cdot x}\,dx$.

Basic problem: Whether $\exists\ \Lambda$ s.t. $E_\Lambda:=\left\{e^{2\pi i\lambda\cdot x}\right\}_{\lambda\in\Lambda}$ forms an orthogonal basis of $L^2(\mu)$?

Background

Classical example: $L^2([0,1])$ has an orthogonal basis $\left\{e^{2\pi i\lambda\cdot x}\right\}_{\lambda\in\mathbb{Z}}$,

i.e., for any $f\in L^2([0,1])$

$$f(x)=\sum_{\lambda\in\mathbb{Z}}c_\lambda(f)e_\lambda(x),\ \text{with } e_\lambda(x)=e^{2\pi i\lambda\cdot x},$$

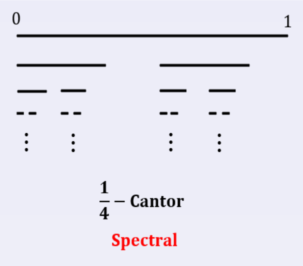
where $c_\lambda(f)=\int_0^1 f(x)e^{-2\pi i\lambda\cdot x}\,dx$.

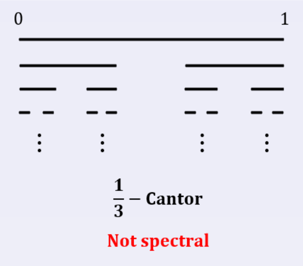
Basic problem: Whether $\exists\ \Lambda$ s.t. $E_\Lambda:=\left\{e^{2\pi i\lambda\cdot x}\right\}_{\lambda\in\Lambda}$ forms an orthogonal basis of $L^2(\mu)$?

Definition 1. Let μ be a Borel probability measure on $\mathbb{R}^d$. The measure μ is called spectral if there exists a countable set $\Lambda\subset\mathbb{R}^d$ such that E_Λ forms an orthogonal basis for $L^2(\mu)$. In this case, we call Λ a spectrum of μ.

Background

Jorgensen and Pedersen(1998) First singular spectral measure





$$\Lambda = \left\{ \sum_{j=0}^{k} 4^j a_j : a_j \in \{0, 1\},\ k \geq 1 \right\}$$

Background

He, Lai and Lau(2013) Spectral measure μ must be of pure type.

- Discrete (finite support), or
- Absolutely continuous, or
- Singularly continuous.

Spectral set and tiling by translations

Definition 2. We say Ω is a spectral set if spectral measure μ is the Lebesgue measure restricted on the set Ω. Usually, we take the measure of Ω to be 1.

Spectral set and tiling by translations

Definition 2. We say Ω is a spectral set if spectral measure μ is the Lebesgue measure restricted on the set Ω. Usually, we take the measure of Ω to be 1.

Definition 3. The set Ω is said to tile $\mathbb{R}^d$ by translations if there exists a discrete set $L \subset \mathbb{R}^d$ such that

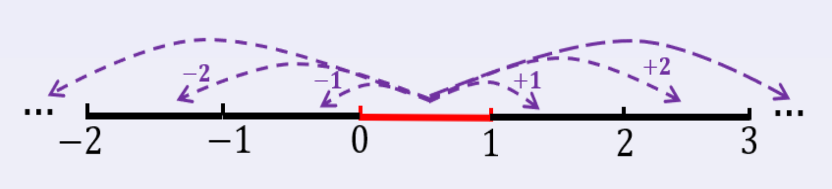
$$\bigcup_{l \in L} (\Omega + l) = \mathbb{R}^d \quad \text{and} \quad m((\Omega + l_1) \cap (\Omega + l_2)) = 0 \ \text{ for all } l_1 \neq l_2 \in L,$$

where $m(\cdot)$ denotes the Lebesgue measure.

Spectral set and tiling by translations

For example

- $\Omega = [0, 1]$

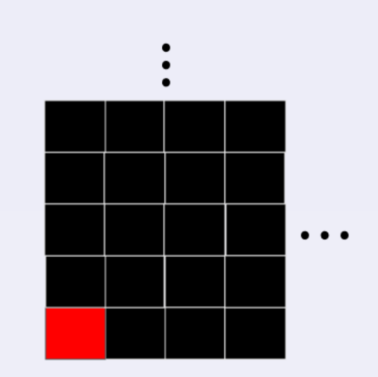


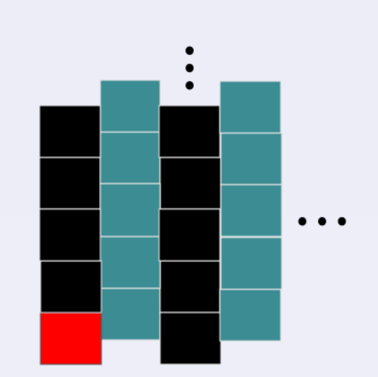
$$\Omega + T = \mathbb{R}$$

where $T = \Lambda = \mathbb{Z}$

Spectral set and tiling by translations

- $\Omega = [0,1]^2$





$\Omega + T_1 = \mathbb{R}^2$ $\qquad\qquad$ $\Omega + T_2 = \mathbb{R}^2$

where $T_1 = \Lambda_1 = \mathbb{Z}^2$ and $T_2 = \Lambda_2 = \begin{pmatrix} 2\mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \bigcup \begin{pmatrix} 2\mathbb{Z}+1 \\ \frac{1}{2}+\mathbb{Z} \end{pmatrix}$

Fuglede Conjecture

Fuglede Conjecture(1974): $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translation.

Fuglede Conjecture

Fuglede Conjecture(1974): $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translation.

Fuglede Conjecture

Counterexamples

- $d \geq 3$, non-tiling spectral set
  - $d \geq 5$ (Tao, 2004)
  - $d = 3, 4$ (Matolcsi, 2004; Kolountzakis and Matolcsi, 2004)

Fuglede Conjecture

Counterexamples

- $d \geq 3$, non-tiling spectral set
    - $d \geq 5$ (Tao, 2004)
    - $d = 3, 4$ (Matolcsi, 2004; Kolountzakis and Matolcsi, 2004)
- $d \geq 3$, tiling not-spectral set
    - $d \geq 5$, (Kolountzakis and Matolcsi, 2004)
    - $d = 3, 4$, (Farkas and Révész, 2004; Farkas, Matolcsi and Móra, 2005).

Fuglede Conjecture

- $d = 1, 2$, conjecture still open in both directions

Fuglede Conjecture

- $d = 1, 2$, conjecture still open in both directions

Positive

- Lattice tile (Fuglede 1974)
- Convex body (Lev and Matolcsi 2022)
- Some cyclic group, for example, $\mathbb{Z}_{p^n}, \mathbb{Z}_p \times \mathbb{Z}_{p^n}, \mathbb{Z}_p^2 \times \mathbb{Z}_q^2 \cdots$

Two line segments

- Fuglede Conjecture holds for an interval.

Question: Does Fuglede Conjecture hold for the union of two intervals?

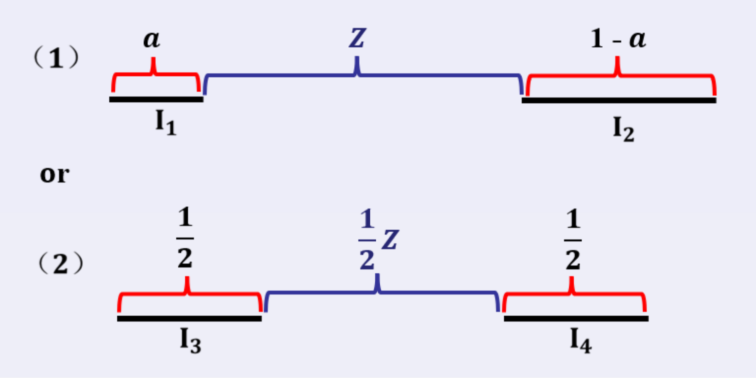
Two line segments

Th A. [Łaba, 2001] Let $\Omega \subset \mathbb{R}$ be a union of two disjoint intervals. Then Ω has a spectrum $\Longleftrightarrow$ it tiles $\mathbb{R}$ by translations.

Two line segments

Th A. [Łaba, 2001] Let $\Omega \subset \mathbb{R}$ be a union of two disjoint intervals. Then Ω has a spectrum $\iff$ it tiles $\mathbb{R}$ by translations.

Essentially all that can happen is this:



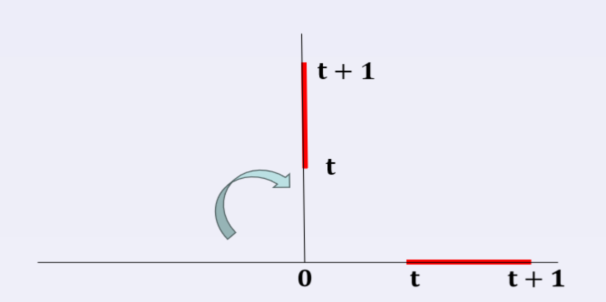
Two line segments

For example $\Omega = [-(t+1), -t] \cup [t, t+1]$ $(t \geq 0)$

spectral $\Longleftrightarrow$ **tile** $\Longleftrightarrow$ $t \in \frac{\mathbb{Z}}{2}$

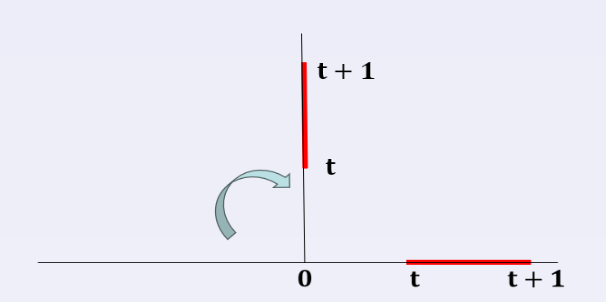
Two line segments

Rotate the left interval to the y-axis. We are seeking a spectrum for singular measure in $\mathbb{R}^2$.



Two line segments

Rotate the left interval to the y-axis. We are seeking a spectrum for singular measure in $\mathbb{R}^2$.



When is it a spectral measure?

Symmetric additive measures

Definition 4. Let μ be a continuous Borel probability measure on $\mathbb{R}$. The symmetric additive measure (SAM) ρ on $\mathbb{R}^2$ is given by

$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \mu),$$

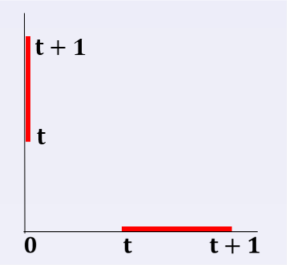
where δ_0 is the Dirac measure at 0. If μ is a Lebesgue measure supported on a unit interval, we call ρ SAML.

Symmetric additive measures

Definition 4. Let μ be a continuous Borel probability measure on $\mathbb{R}$. The symmetric additive measure (SAM) ρ on $\mathbb{R}^2$ is given by

$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \mu),$$

where δ_0 is the Dirac measure at 0. If μ is a Lebesgue measure supported on a unit interval, we call ρ SAML.

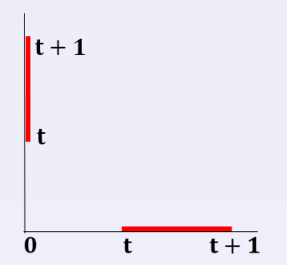


Symmetric additive measures

Definition 4. Let μ be a continuous Borel probability measure on $\mathbb{R}$. The symmetric additive measure (SAM) ρ on $\mathbb{R}^2$ is given by

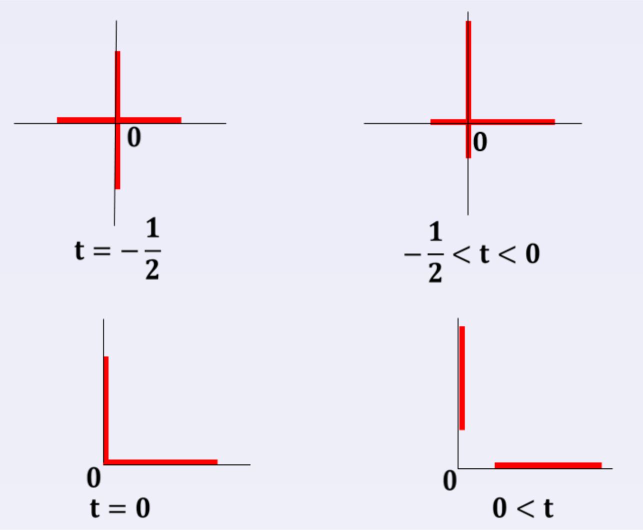
$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \mu),$$

where δ_0 is the Dirac measure at 0. If μ is a Lebesgue measure supported on a unit interval, we call ρ SAML.



When is SAML a spectral measure?

By the symmetry, we consider only the cases $t \geq -\frac{1}{2}$.



Th B. [Lai, Liu and Prince (2021); Ai, Lu and Zhou (2023)] Let ρ be SAML, then the following two statements hold:

(1). If $-\frac{1}{2} < t < 0$ and $2t + 1 = \frac{1}{a}$, where $a > 1$ is a positive integer, then ρ is not spectral.

(2). If $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$, then ρ is a spectral measure $\Longleftrightarrow$ $t \in \frac{\mathbb{Z}}{2}$.
In this case, ρ has a unique spectrum of the form

$$\Lambda = \{(\lambda, -\lambda) : \lambda \in \Lambda_0\},$$

where Λ_0 is the spectrum of the Lebesgue measure supported on $[-t-1, -t] \cup [t, t+1]$.

Our goal is to prove

$$\begin{cases} (1).\text{When } -\frac{1}{2} < t < 0, \rho \text{ is not a spectral measure.} \\ \\ (2).\text{When } t \notin \mathbb{Q}, \rho \text{ is not a spectral measure.} \end{cases}$$

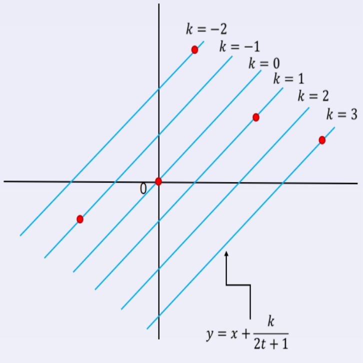
SAML

Lem 1. Let ρ be SAML. If $t \neq -\frac{1}{2}$ and ρ is spectral with spectrum $0 \in \Lambda \subseteq \mathbb{R}^2$, then for every $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ there exists an integer $k(\lambda)$ such that

$$\lambda_2 - \lambda_1 = \frac{k(\lambda)}{2t+1}.$$

Lem 1. Let ρ be SAML. If $t \neq -\frac{1}{2}$ and ρ is spectral with spectrum $0 \in \Lambda \subseteq \mathbb{R}^2$, then for every $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ there exists an integer $k(\lambda)$ such that

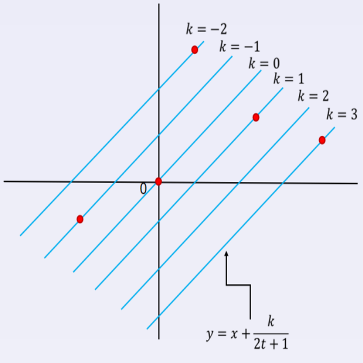
$$\lambda_2 - \lambda_1 = \frac{k(\lambda)}{2t+1}.$$



SAML

Lem 1. Let ρ be SAML. If $t \neq -\frac{1}{2}$ and ρ is spectral with spectrum $0 \in \Lambda \subseteq \mathbb{R}^2$, then for every $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ there exists an integer $k(\lambda)$ such that

$$\lambda_2 - \lambda_1 = \frac{k(\lambda)}{2t+1}.$$



Proof :

- For any $\lambda_1 \neq \lambda_2 \in \Lambda$,

$$0 = \langle e^{2\pi i \lambda_1 x}, e^{2\pi i \lambda_2 x} \rangle_{L^2(\rho)}$$
$$= \int e^{2\pi i(\lambda_1 - \lambda_2)x} d\rho = \hat{\rho}(\lambda_2 - \lambda_1)$$

- $\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\}$
$$\subset \{x : \hat{\rho}(x) = 0\}$$

Proof of Lem 1

- Zeros set of $\hat{\rho}$

$$\{\lambda : \hat{\rho}(\lambda) = 0\} = \left\{(\lambda_1, \lambda_2) : e^{\pi i(\lambda_1 - \lambda_2)(2t+1)} \frac{\sin \pi\lambda_1}{\pi\lambda_1} = -\frac{\sin \pi\lambda_2}{\pi\lambda_2}\right\}$$

- So the exponential factor must be real, for vanishing, which gives

$$\Lambda \subset \left\{(\lambda_1, \lambda_2) :\ \lambda_2 - \lambda_1 \in \frac{1}{2t+1}\mathbb{Z}\right\}$$

Intersecting case

Th 1. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $-\frac{1}{2} < t < 0$, then ρ is not spectral.

Intersecting case

Th 1. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $-\frac{1}{2} < t < 0$, then ρ is not spectral.

Proof: **(By contradiction)** Suppose $0 \in \Lambda$ is a spectrum.

(1). $f(x) \overset{L^2}{=} \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle e_\lambda(x) \quad$ for any $f \in L^2(\rho)$

Intersecting case

Th 1. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $-\frac{1}{2} < t < 0$, then ρ is not spectral.

Proof: **(By contradiction)** Suppose $0 \in \Lambda$ is a spectrum.

(1). $f(x) \overset{L^2}{=} \sum_{\lambda\in\Lambda} \langle f, e_\lambda \rangle e_\lambda(x) \quad$ for any $f \in L^2(\rho)$

(2). $f(x+T) \overset{L^2}{=} f(x)$, where $T = (2t+1, -2t-1)$ (By Lemma 1)

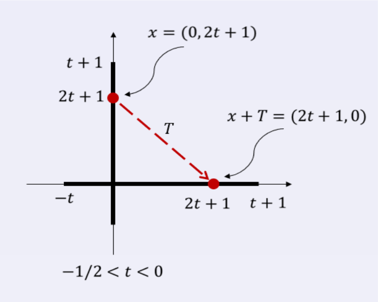
Intersecting case

Th 1. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $-\frac{1}{2} < t < 0$, then ρ is not spectral.

Proof: **(By contradiction)** Suppose $0 \in \Lambda$ is a spectrum.

(1). $f(x) \overset{L^2}{=} \sum_{\lambda\in\Lambda}\langle f, e_\lambda\rangle e_\lambda(x)$ for any $f \in L^2(\rho)$

(2). $f(x+T) \overset{L^2}{=} f(x)$, where $T = (2t+1, -2t-1)$ (By Lemma 1)



Proof of Theorem 1

(3). Construct a function $f(x) \in L^2(\rho)$ s.t. for all $x \in supp(\rho)$

$$\begin{cases} f(x) \overset{\color{red}{Pointwise}}{=} \sum_{\lambda\in\Lambda}\langle f, e_\lambda\rangle e_\lambda(x) \\ f(x+T) \overset{\color{red}{Pointwise}}{=} f(x) \end{cases}$$

Proof of Theorem 1

(3). Construct a function $f(x) \in L^2(\rho)$ s.t. for all $x \in supp(\rho)$

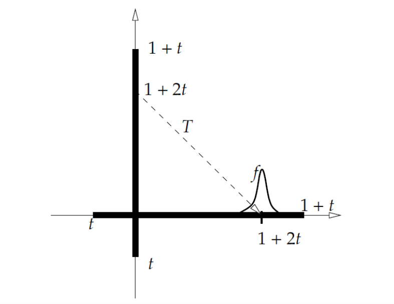
$$\begin{cases} f(x) \overset{\color{red}{Pointwise}}{=} \sum_{\lambda\in\Lambda}\langle f, e_\lambda\rangle e_\lambda(x) \\ f(x+T) \overset{\color{red}{Pointwise}}{=} f(x) \end{cases}$$

- Vertical segment 0

$$f(0, y) = 0 \text{ for all } y \in \mathbb{R}$$

- Horizontal segment

$$\begin{cases} f(2t+1, 0) = 1 \\ \textit{smooth function} \\ \textit{support close to } (1+2t, 0) \end{cases}$$



Proof of Theorem 1

$\Longrightarrow \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle| < \infty$ for all $x \in \mathbb{R}^2$ (from smoothness)

$\Longrightarrow 1 = f(2t+1, 0) = f(0, 2t+1) = 0$, contradiction!

Case of irrational t

Th 2. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $t \notin \mathbb{Q}$, then ρ is not spectral.

Case of irrational t

Th 2. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $t \notin \mathbb{Q}$, then ρ is not spectral.

Proof: **(By contradiction)** Suppose $0 \in \Lambda$ is a spectrum.

Case of irrational t

Th 2. [M. Kolountzakis, S. Wu(2025)] Let ρ be SAML. If $t \notin \mathbb{Q}$, then ρ is not spectral.

Proof: (By contradiction) Suppose $0 \in \Lambda$ is a spectrum.

Lem 2. (1). On each line of the form

$$y - x = \frac{k}{2t+1} \quad \text{for some } k \in \mathbb{Z},$$

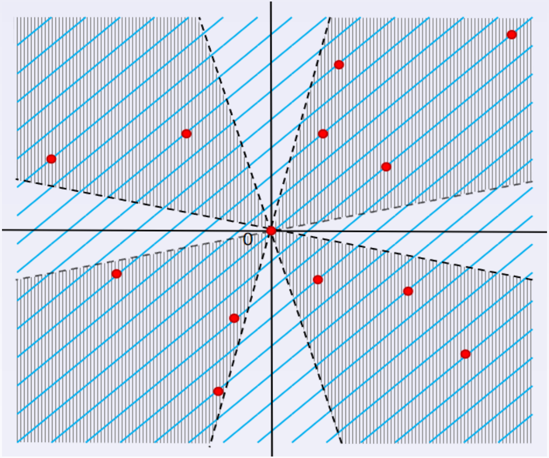
there is at most one point of Λ.

(2). There is a constant $K > 1$ such that

$$K^{-1}|\lambda_1| \leq |\lambda_2| \leq K|\lambda_1|$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Lambda$.

Proof of Theorem 2



Remark. [Lai, Liu and Prince (2021)] At most one point on any vertical or horizontal line through Λ.

Proof of Theorem 2

Lem 3. Let Λ_x, Λ_y be the projections of Λ on the $x, y - axis$. We have

$$2 = \sum_{\lambda \in \Lambda_x} |\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2(x-\lambda) \quad \text{and} \quad 2 = \sum_{\lambda \in \Lambda_y} |\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2(x-\lambda).$$

In this case, we call $|\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2 + \Lambda_x$ a tiling of $\mathbb{R}$ at level 2.

Proof of Theorem 2

Lem 3. Let Λ_x, Λ_y be the projections of Λ on the $x, y-axis$. We have

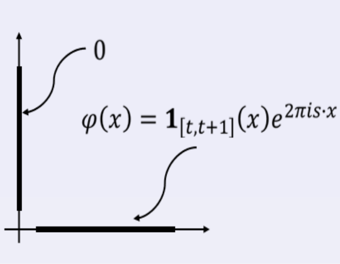
$$2 = \sum_{\lambda \in \Lambda_x} |\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2(x-\lambda) \quad \text{and} \quad 2 = \sum_{\lambda \in \Lambda_y} |\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2(x-\lambda).$$

In this case, we call $|\widehat{\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}}|^2 + \Lambda_x$ a tiling of $\mathbb{R}$ at level 2.

Proof :

- $\|f\|_{L^2(\rho)} = \sum_{\lambda \in \Lambda} \left|\langle f, e_\lambda \rangle_{L^2(\rho)}\right|^2$

- $f(x,y) = \begin{cases} 0, & x = 0 \\ \varphi(x), & y = 0 \end{cases}$



Proof of Theorem 2

Lem 4. There are finitely many different gaps among successive points in Λ_x and Λ_y. In this case, we call Λ_x, Λ_y finite complexity.

Lem 4. There are finitely many different gaps among successive points in Λ_x and Λ_y. In this case, we call Λ_x, Λ_y finite complexity.

Proof :

- Only need to consider Λ_x.

  Writing the set $\Lambda_x = \{\lambda_1^n : n \in \mathbb{Z}\}$ with

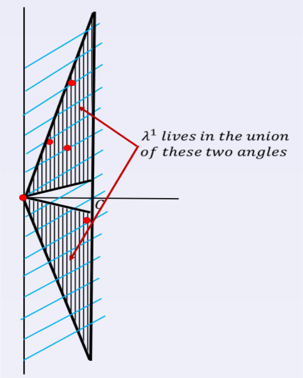
$$\cdots \leq \lambda_1^{-1} \leq \lambda_1^0 = 0 \leq \lambda_1^1 \leq \lambda_1^2 < \cdots$$

and $\Lambda = \{\lambda^n = (\lambda_1^n, \lambda_2^n) : n \in \mathbb{Z}\}$

To prove the set $\{\lambda_1^{n+1} - \lambda_1^n\}_{n \in \mathbb{Z}}$ finitely many different elements.

Proof of Theorem 2

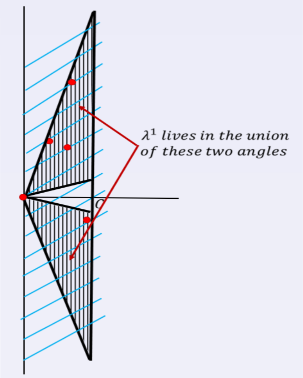
- By the tiling property (Lemma 3), there is a constant C such that $\lambda_1^1 < C$
- λ^1 lives in the union of these two angles



Proof of Theorem 2

- By the tiling property (Lemma 3), there is a constant C such that $\lambda_1^1 < C$

- λ^1 lives in the union of these two angles



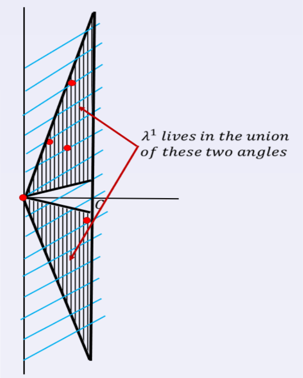
- Translate Λ by $-\lambda^n$, we have

$$\lambda^{n+1} - \lambda^n \text{ lives again in the union of these two angles}$$

Proof of Theorem 2

- By the tiling property (Lemma 3), there is a constant C such that $\lambda_1^1 < C$

- λ^1 lives in the union of these two angles



- Translate Λ by $-\lambda^n$, we have

$$\lambda^{n+1} - \lambda^n \text{ lives again in the union of these two angles}$$

- On each blue line, the zeros of $\hat{\rho}$ are a discrete set

$\Longrightarrow \{\lambda_1^{n+1} - \lambda_1^n\}_{n\in\mathbb{Z}}$ can take only finitely many values.

Proof of Theorem 2

Th C. [Kolountzakis and Lev(2016)] If

(1). $f + \Lambda$ is a tiling of $\mathbb{R}$ at some level l.

(2). Λ has finite complexity and spectral gap.

Then $\Lambda = \Lambda + T$ for some positive $T \in \mathbb{R}$. In other words, Λ is periodic set.

Proof of Theorem 2

Th C. [Kolountzakis and Lev(2016)] If

(1). $f + \Lambda$ is a tiling of $\mathbb{R}$ at some level l.

(2). Λ has finite complexity and spectral gap.

Then $\Lambda = \Lambda + T$ for some positive $T \in \mathbb{R}$. In other words, Λ is periodic set.

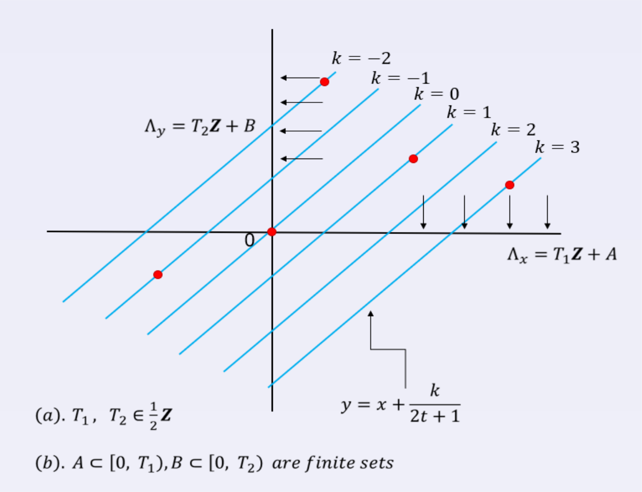
Combining Lemmas 3 and 4 with Theorem C, we have

Lem 5. There are positive $T_1, T_2 \in \frac{1}{2}\mathbb{Z}$ and some finite sets $A \subseteq [0, T_1)$ and $B \subseteq [0, T_2)$ such that

$$\Lambda_x = T_1\mathbb{Z} + A \quad \text{and} \quad \Lambda_y = T_2\mathbb{Z} + B$$

Proof of Theorem 2

- Distribution of Λ



Proof of Theorem 2

- For any $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, there are $k, m, n \in \mathbb{Z}$, $a \in A$ and $b \in B$ such that

$$\begin{cases} \lambda_2 - \lambda_1 = \frac{k}{2t+1} \\ \lambda_1 = mT_1 + a \\ \lambda_2 = nT_2 + b \end{cases}$$

Proof of Theorem 2

- For any $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, there are $k, m, n \in \mathbb{Z}$, $a \in A$ and $b \in B$ such that

$$\begin{cases} \lambda_2 - \lambda_1 = \frac{k}{2t+1} \\ \lambda_1 = mT_1 + a \\ \lambda_2 = nT_2 + b \end{cases}$$

- $\{2\lambda_2 - 2\lambda_1\} = \{2b - 2a\} = \{\frac{2k}{2t+1}\}$, where $\{\cdot\}$ denotes fractional part

Proof of Theorem 2

- For any $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, there are $k, m, n \in \mathbb{Z}$, $a \in A$ and $b \in B$ such that

$$\begin{cases} \lambda_2 - \lambda_1 = \frac{k}{2t+1} \\ \lambda_1 = mT_1 + a \\ \lambda_2 = nT_2 + b \end{cases}$$

- $\{2\lambda_2 - 2\lambda_1\} = \{2b - 2a\} = \{\frac{2k}{2t+1}\}$, where $\{\cdot\}$ denotes fractional part

- $\begin{cases} \{2b - 2a\} \ \text{finitely many values} \\ \{\frac{2k}{2t+1}\} \ \text{all values are different since } t \text{ is irrational} \end{cases}$

$\Longrightarrow$ Contradiction

SAML

- In summary, if $t \neq -\frac{1}{2}$, then ρ is a spectral measure $\Longleftrightarrow t \in \frac{\mathbb{Z}}{2}$.

SAML

- In summary, if $t \neq -\frac{1}{2}$, then ρ is a spectral measure $\Longleftrightarrow t \in \frac{\mathbb{Z}}{2}$.
- **Lu(2025); Kolountzakis and Lai(2025)**] If $t = -\frac{1}{2}$ (Plus type), then ρ is not a spectral measure.

- In summary, if $t \neq -\frac{1}{2}$, then ρ is a spectral measure $\Longleftrightarrow t \in \frac{\mathbb{Z}}{2}$.

- **Lu(2025); Kolountzakis and Lai(2025)]** If $t = -\frac{1}{2}$ (Plus type), then ρ is not a spectral measure.

Th 3. [Lai, Liu and Prince (2021); Ai, Lu and Zhou (2023); Kolountzakis, Wu(2025); Lu(2025); Kolountzakis and Lai(2025)] If ρ is SAML, then ρ is a spectral measure $\Longleftrightarrow t \in \frac{\mathbb{Z}}{2} \setminus \{-\frac{1}{2}\}$.

In this case, ρ has a unique spectrum of the form

$$\Lambda = \{(\lambda, -\lambda) : \lambda \in \Lambda_0\},$$

where Λ_0 is the spectrum of the Lebesgue measure supported on $[-t-1, -t] \cup [t, t+1]$.

Projection measure

L : a straight line through the origin

u : a unit vector along L

$u^\perp$: the orthogonal subspace to L.

π_L : the orthogonal projection onto line L

$$\pi_L(v) = t \ \text{ for any } \ v \in tu + u^\perp$$

ρ : a Borel measure on $\mathbb{R}^2$

$\pi_L\rho$: a projection measure on $\mathbb{R}$

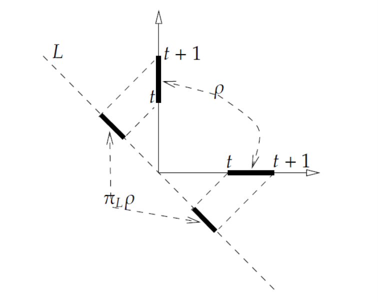
$$\pi_L\rho(E) = \rho(Eu + u^\perp), \ E \subseteq \mathbb{R}$$

Projection measure

For example:

- $L: y = -x$
- $Supp(\pi_L \rho)$

$= \frac{1}{\sqrt{2}} \left((-(t+1), -t) \cup (t, t+1) \right)$



Projection measure

Th 3. [M. Kolountzakis, S. Wu(2025)] If
(1) ρ is a probability measure on $\mathbb{R}^2$ whose support is a finite union of line segments;
(2) L is a straight line through the origin such that the orthogonal projection π_L onto L is one-to-one ρ-almost everywhere.

Then ρ has a spectrum $\Lambda u \subseteq L \iff \pi_L\rho$ has spectrum $\Lambda \subseteq \mathbb{R}$, where u is a unit vector along L.

Projection measure

Th 3. [M. Kolountzakis, S. Wu(2025)] If
(1) ρ is a probability measure on $\mathbb{R}^2$ whose support is a finite union of line segments;
(2) L is a straight line through the origin such that the orthogonal projection π_L onto L is one-to-one ρ-almost everywhere.

Then ρ has a spectrum $\Lambda u \subseteq L \iff \pi_L \rho$ has spectrum $\Lambda \subseteq \mathbb{R}$, where u is a unit vector along L.

Proof : Any function $f(x)$ on suppρ can be written as

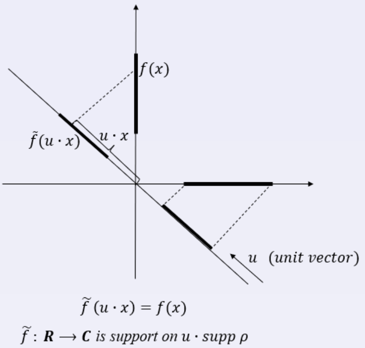
$$f(x) = \widetilde{f}(u \cdot x) \quad \text{for } \rho - \text{almost all } x,$$

where $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ is supported on $u \cdot \mathrm{supp}\rho$.

Proof of Theorem 3

- $\| f \|_{L^2(\rho)} = \| \widetilde{f} \|_{L^2(\pi_L \rho)}$

- $e^{2\pi i \lambda u \cdot x} = e^{2\pi i \lambda (u \cdot x)}$
  $= e^{2\pi i \lambda \cdot \pi_L(x)}$



-

$$\|f\|^2_{L^2(\rho)} = \sum_{\lambda \in \Lambda u} \left| \langle f, e_\lambda \rangle_{L^2(\rho)} \right|^2$$

$$\Updownarrow$$

$$\|\widetilde{f}\|^2_{L^2(\pi_L \rho)} = \sum_{\lambda \in \Lambda} \left| \langle \widetilde{f}, e_\lambda \rangle_{L^2(\pi_L \rho)} \right|^2$$

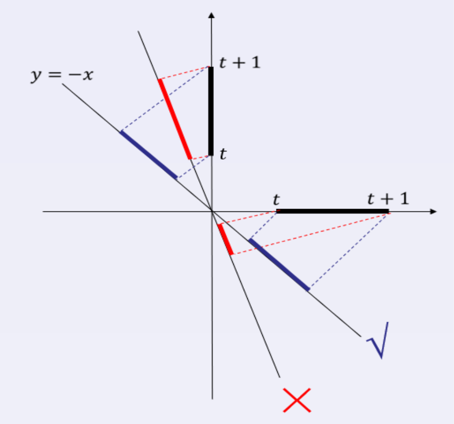
Projection measure

Th D. [Dutkay and Lai (2014)] If an absolutely continuous measure μ on $\mathbb{R}^d$ is a spectral, then it is a constant multiple of Lebesgue measure on its support.

Projection measure

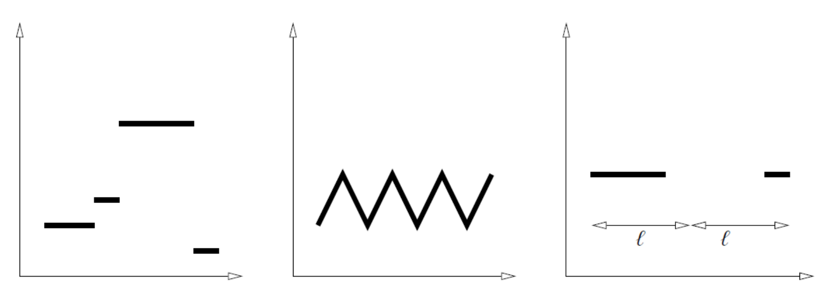
Th D. [Dutkay and Lai (2014)] If an absolutely continuous measure μ on $\mathbb{R}^d$ is a spectral, then it is a constant multiple of Lebesgue measure on its support.

For example



Projection measure

For example



These measures all have a spectrum contained in the x-axis

Thanks!